American University of Beirut
Math 102
Quiz I (Summer 2013)

Time 60 minutes

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Solution

1- Let $f(x)=\sqrt{x-7}$.

a. Find a formula for $f^{-1}(x)$.

3 pts)

$y=\sqrt{x-7} \longrightarrow y^2=x-7 \longrightarrow x=y^2+7$

The inverse function is $y=x^2+7$

$f^{-1}(x)=x^2+7$

b. Determine the domain and range of f and f^{-1}

3 pts)

Domain of $f=[7,+\infty)$ = Range of f^{-1}

Range of $f=[0,+\infty)$ = domain of f^{-1}

2- Find the following limits

4 pts)

- $\lim_{x\to\infty} e^{-x}\ln(x^2+4)$ $= \lim_{x\to\infty} \frac{\ln(x^2+4)}{e^x} \left[\frac{\infty}{\infty}\right]$

$0\times\infty$

$\overset{H.R.}{=} \lim_{x\to\infty} \frac{\frac{2x}{x^2+4}}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x(x^2+4)} \left[\frac{\infty}{\infty}\right]$

$\overset{H.R.}{=} \lim_{x\to\infty} \frac{2}{\underset{\infty}{e^x(x^2+4)}+\underset{\infty}{e^x(2x)}} = 0$

(5 pts)

- $\lim_{x\to 0}(e^x+x)^{\frac{1}{x}}$

$(\sim 1)^\infty$

let $y=(e^x+x)^{\frac{1}{x}} \longrightarrow \ln y=\frac{1}{x}\ln(e^x+x)$

$\lim_{x\to 0}\ln y = \lim_{x\to 0} \underset{\infty}{\frac{1}{x}} \underset{-\infty}{\ln(e^x+x)} = -\infty$

$\therefore \lim_{x\to 0} y = e^{-\infty}=0$

(5pts)

- $\lim_{x\to 0^+}\frac{2\ln x}{\ln(\sin x)}$ $\left[\frac{\infty}{\infty}\right]$

$\overset{H.R}{=} \lim_{x\to 0^+} \frac{2/x}{\frac{1}{\sin x}\cos x} = \lim_{x\to 0} \frac{2}{x}\cdot\frac{\sin x}{\cos x} \left[\frac{0}{0}\right] =$

$\overset{H.R}{=} \lim_{x\to 0} \frac{2\cos x}{\underset{1}{\cos x} - \underset{0}{x\sin x}} = 2$

4. Find the derivative $\dfrac{dy}{dx}$:

pts)

- $y = 3^{5x} + \tan^{-1}(2x+7) + \ln 2$

$$y' = 3^{5x} \ln 3 \cdot 5 + \frac{2}{1+(2x+7)^2} + 0$$

(3 pts)

- $y = (\ln x)^{\pi} - x^{e}$

$$y' = \pi (\ln x)^{\pi - 1} \cdot \frac{1}{x} - e x^{e-1}$$

(5 pts)

- $y = \cos^{-1}[\ln(3x+5)]$

$$y' = \frac{-1}{\sqrt{1-[\ln(3x+5)]^2}} \cdot [\ln(3x+5)]' = \frac{-1}{\sqrt{1-[\ln(3x+5)]^2}} \cdot \frac{3}{3x+5}$$

(6 pts)

- $y = (\sin x)^{\cos x}$

$$\ln y = \ln(\sin x)^{\cos x} \rightarrow \ln y = \cos x \ln(\sin x)$$

$$\frac{y'}{y} = -\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}$$

$$y' = \left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)(\sin x)^{\cos x}$$

5. Simplify the following expressions

(3 pts)

- $e^{4\ln 3 - 3\ln 4}$ $= e^{\ln 3^4 - \ln 4^3} = e^{\ln \frac{3^4}{4^3}} = \dfrac{3^4}{4^3}$

(5 pts)

- $\log_3\left[\ln\left(\sqrt{7+e^3}+\sqrt{7}\right)+\ln\left(\sqrt{7+e^3}-\sqrt{7}\right)\right]$

$$\log_3\left[\ln\left(\sqrt{7+e^3}+\sqrt{7}\right)\left(\sqrt{7+e^3}-\sqrt{7}\right)\right]$$

$$\log_3\left[\ln\left(\cancel{7}+e^3-\cancel{7}\right)\right] = \log_3\left(\ln e^3\right) = \log_3 3 = 1$$

6. Evaluate the following integral

ts)

- $\int_1^e x^{(\ln 2)-1}\,dx$ $= \left.\dfrac{x^{\ln 2}}{\ln 2}\right]_1^e = \dfrac{1}{\ln 2}\left[e^{\ln 2} - 1^{\ln 2}\right] =$

$\dfrac{1}{\ln 2}\left[2 - 1\right] = \dfrac{1}{\ln 2}$

pts)

- $\int \dfrac{5^{\frac{-1}{x^2}}}{x^3}\,dx$ let $-\dfrac{1}{x^2} = u \to -x^{-2} = u$

$\to +2x^{-3}dx = du$

$\dfrac{1}{x^3}\,dx = \dfrac{1}{2}\,du$

$\dfrac{1}{2}\int 5^u\,du = \dfrac{1}{2}\dfrac{5^u}{\ln 5} + C$

$= \dfrac{1}{2}\dfrac{5^{-1/x^2}}{\ln 5} + C$

pts)

- $\int \cos^3 x\, \sin^2 x\, dx$ $= \int \cos^2 x \sin^2 x \cos x\, dx$

$= \int (1-\sin^2 x)\sin^2 x \cos x\, dx$ let $\sin x = u$, $\cos x\,dx = du$

$= \int (1-u^2)u^2\,du = \int (u^2 - u^4)\,du$

$= \dfrac{u^3}{3} - \dfrac{u^5}{5} + C = \dfrac{\sin^3 x}{3} - \dfrac{\sin^5 x}{5} + C$

5 pts)

- $\int \cos 3x\, \sin 5x\, dx$ $= \int \sin 5x \cos 3x\, dx$

si co

$= \dfrac{1}{2}\int (\sin 2x + \sin 8x)\,dx = \dfrac{1}{2}\left[-\dfrac{\cos 2x}{2} - \dfrac{\cos 8x}{8}\right] + C$

$= -\dfrac{1}{2}\left(\dfrac{\cos 2x}{2} + \dfrac{\cos 8x}{8}\right) + C$

pts)

- $\int (x^2 - 5x)\sin 2x\, dx$

$$= -(x^2-5x)\frac{\cos 2x}{2} + (2x-5)\frac{\sin 2x}{4} + \frac{2\cos 2x}{8} + C$$

f and derivatives		g and integrals
x^2-5x	$+$	$\sin 2x$
$2x-5$	$-$	$-\frac{\cos 2x}{2}$
2	$+$	$-\frac{\sin 2x}{4}$
0		$\frac{\cos 2x}{8}$

pts)

- $\int x^2 \ln x \, dx$

D	I
$\ln x$	$x^2 dx$
$\frac{1}{x} dx$	$\frac{x^3}{3}$

$-\int$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + C$$

pts)

- $\int \frac{2x+1}{x^2-7x+12}\, dx \quad = \int \frac{2x+1}{(x-3)(x-4)}\, dx$

$$\frac{2x+1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

$$2x+1 = A(x-4) + B(x-3)$$

$x=4 \longrightarrow \boxed{9 = B}$

$x=+3 \longrightarrow 7 = -A \longrightarrow \boxed{A=-7}$

$$= -7\int \frac{dx}{x-3} + 9\int \frac{dx}{x-4} = -7\ln|x-3| + 9\ln|x-4| + C$$

pts)

- $\int \frac{dy}{y^2-4y+5}$ $= \int \frac{dy}{(y^2-4y+4)+1} = \int \frac{dy}{(y+2)^2+1}$

let $y+2=u$

$dy = du$ $\quad = \int \frac{du}{u^2+1} = \tan^{-1} u + C$

$= \tan^{-1}(y+2) + C$

pts)

- $\int \frac{x^2}{\sqrt{25-4x^2}}\,dx$ let $2x = 5\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$a^2 - u^2$

$5^2 - (2x)^2$

$x = \frac{5}{2}\sin\theta$

$dx = \frac{5}{2}\cos\theta\, d\theta$

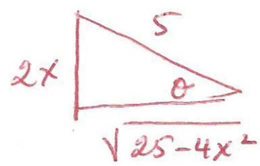
$$\frac{5}{2}\int \frac{\left(\frac{5}{2}\right)^2 \sin^2\theta}{\sqrt{25-25\sin^2\theta}} \cos\theta\, d\theta = \left(\frac{5}{2}\right)^3 \cdot \frac{1}{5} \int \frac{\sin^2\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \left(\frac{5}{2}\right)^3 \cdot \frac{1}{5} \int \frac{\sin^2\theta \cos\theta}{|\cos\theta|}\, d\theta = \left(\frac{5}{2}\right)^3 \cdot \frac{1}{5} \int \frac{\sin^2\theta\, \cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta$$

$$= \left(\frac{5}{2}\right)^3 \cdot \frac{1}{5} \int \frac{1-\cos 2\theta}{2}\, d\theta = \left(\frac{5}{2}\right)^3 \cdot \frac{1}{5} \cdot \frac{1}{2} \int (1-\cos 2\theta)\, d\theta$$

$2\sin\theta\cos\theta$

$$= \left(\frac{5}{2}\right)^3 \cdot \frac{1}{10} \left[\theta - \frac{\sin 2\theta}{2}\right] + C$$



$$= \left(\frac{5}{3}\right)^3 \cdot \frac{1}{10} \left[\sin^{-1}\frac{2x}{5} - \frac{2x}{5} \cdot \frac{\sqrt{25-4x^2}}{5}\right] + C$$